

The Hall effect in accretion flows

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ABSTRACT

Magnetic diffusion in accretion flows changes the structure and angular momentum of the accreting material. We present two power law similarity solutions for flattened accretion flows in the presence of magnetic diffusion: a secularly-evolving Keplerian disc and a magnetically-diluted free fall onto the central object. The influence of Hall diffusion on the solutions is evident even when this is small compared to ambipolar and Ohmic diffusion, as the surface density, accretion rate and angular momentum in the flow all depend upon the product $\eta_H(\mathbf{B} \cdot \boldsymbol{\Omega})$, and the inclusion of Hall diffusion may be the solution to the magnetic braking catastrophe of star formation simulations.

Key words: accretion, accretion discs – diffusion – magnetic fields – MHD – stars: formation.

1 INTRODUCTION

Accretion flows onto protostars are flattened and braked by magnetic fields. The transportation of angular momentum from the flow to the envelope is facilitated by the bending of the magnetic field lines and inhibited by the drift of those lines against the flow¹. In magnetohydrodynamical (MHD) simulations of star formation it has been shown that it is possible to remove all of the angular momentum from the accretion flow using a weak field, with the consequence that no accretion disc forms (e.g. Price & Bate 2007; Mellon & Li 2008). Magnetic diffusion reduces the braking so that a protostellar disc forms (e.g. Krasnopolsky & Königl 2002, hereafter KK02; Mellon & Li 2009); these discs are precursors to those in which planet formation occurs.

The amount of magnetic braking affecting an accretion flow depends on the coupling of the largely neutral medium to the magnetic field, which is facilitated by collisions between the neutral and charged particles that transmit the Lorentz force to neutrals. In most simulations the magnetic field behaviour has been approximated by ideal MHD (where the magnetic field is frozen into the neutral particles), however this approximation breaks down as the density in the disc increases. The relative drifts of different charged species with respect to the neutral particles delineate three conductivity regimes. *Ohmic (resistive)* and *ambipolar diffusion*, which behave similarly in a thin disc (to a first order approximation), have been shown to reduce the effectiveness of magnetic braking (Shu et al. 2006; Machida et al. 2007; Mellon & Li 2009). *Hall diffusion* occurs when the degree of

coupling between the different charged species and the neutrals varies, and is expected to dominate in large regions in accretion flows (Sano & Stone 2002; Wardle 2007). For certain field configurations Hall diffusion can in principle cause the magnetic field to be accreted faster than the neutral fluid (Wardle & Ng 1999; Braiding & Wardle 2012).

The role of Hall diffusion in star formation and accretion discs is neither fully understood nor explored due to the difficulty of performing numerical simulations (although this is starting to change, e.g. Li et al. 2011; Krasnopolsky et al. 2011), highlighting the need for simpler analytic models that demonstrate the importance of the Hall effect. In this paper we present two power law similarity solutions to the MHD equations for an isothermal thin disc with Hall, ambipolar and Ohmic diffusion: one in which the disc is Keplerian and a second in which the collapsing material spirals onto the central object without disc formation. We discuss the implications and the physics controlling these remarkable solutions.

2 FORMULATION

The magnetohydrodynamic equations for an isothermal system are given by

$$\rho \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (1)$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P - \rho \nabla \Phi + \mathbf{J} \times \mathbf{B}, \quad (2)$$

$$\nabla^2 \Phi = 4\pi G \rho, \quad (3)$$

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¹ This is only strictly true for ambipolar and Ohmic diffusion.

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \nabla \times \left[\eta (\nabla \times \mathbf{B}) + \eta_H (\nabla \times \mathbf{B}) \times \hat{\mathbf{B}} + \eta_A (\nabla \times \mathbf{B}) \right], \quad (5)$$

where ρ is the gas density, \mathbf{V} the velocity field, P the mid-plane gas pressure, Φ the gravitational potential (defined as $\mathbf{g} = -\nabla \Phi$ where \mathbf{g} is the gravitational field), G the gravitational constant, \mathbf{B} the magnetic field and \mathbf{J} the current density defined as $\mathbf{J} = c(\nabla \times \mathbf{B})/4\pi$ where c is the speed of light. The diffusion coefficients for the Ohmic, Hall and ambipolar terms in the induction equation are η , η_H and η_A respectively.

We adopt cylindrical coordinates and assume that the disc is axisymmetric and thin. The velocity field, $\mathbf{V}(r)$, has both radial and azimuthal field components, the latter giving rise to the specific angular momentum $J = rV_\phi$, as the material is assumed to settle rapidly to the disc midplane. We assume that the disc is threaded by an open magnetic field that is symmetric about the midplane, defining B_z as the vertical field component at the midplane and B_{rs} and $B_{\phi s}$ as the components at the surface $z = H$, where $H(r)$ is the half-thickness of the disc. We neglect any mass loss due to a disc wind. As the disc is isothermal, the pressure at the midplane is given by $P = \rho c_s^2$, where c_s is the constant isothermal sound speed, typically taken to be 0.19 km s^{-1} .

Equations 1–5 are vertically-averaged as in Braiding & Wardle (2012, following KK02) by assuming the disc is thin and integrating over z , which allows us to discard any terms of order H/r . The density, radial velocity, azimuthal velocity and radial gravity are approximated as being constant with height, as are the quantities η , η_H/B and η_A/B^2 . The radial and azimuthal magnetic field components are taken to scale linearly with height, so that their values at the disc surface (B_{rs} and $B_{\phi s}$) may be used to characterise the field.

The equations are further simplified by employing monopole expressions for B_{rs} and g_r :

$$B_{rs} = \frac{\Psi(r, t)}{2\pi r^2} \quad (6)$$

$$\text{and } g_r = -\frac{GM(r, t)}{r^2}, \quad (7)$$

where M and Ψ are the mass and magnetic flux enclosed within a radius r . These describe the disc sufficiently well that a more complicated iterative method of calculating the field and gravity is unnecessary (Contopoulos et al. 1998).

The azimuthal field component is calculated by balancing the torques on the disc from rotation and magnetic braking, which transports angular momentum from the disc to the low-density external medium into which the field is frozen (Basu & Mouschovias 1994). Assuming that the background angular frequency is small and that the external Alfvén wave speed is constant and parameterised by $V_{A, \text{ext}} = c_s/\alpha$, the azimuthal field component is then

$$B_{\phi s} = -\min \left[\frac{\Psi \alpha}{\pi r^2 c_s} \left[\frac{J}{r} - \frac{\eta_H B_{rs}}{HB} \right] \left[1 + \frac{\Psi \alpha}{\pi r^2 c_s} \frac{\eta_A B_z}{B^2 H} \right]^{-1}; \delta B_z \right] \quad (8)$$

(Braiding & Wardle 2012). The field is capped as a way of representing the many magnetohydrodynamical instabilities and processes that might prevent the azimuthal field component from exceeding the poloidal component such as internal kinks, turbulence and the magnetorotational instability (MRI). The cap $\delta = 1$ corresponds to the typical value of $|B_{\phi s}|/B_z$ obtained when the vertical angular momentum transport is dominated by a centrifugally-driven disc wind (KK02).

The vertically-averaged equations are then:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma V_r) = 0, \quad (9)$$

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} = g_r - \frac{c_s^2}{\Sigma} \frac{\partial \Sigma}{\partial r} + \frac{B_z B_{rs}}{2\pi \Sigma} + \frac{J^2}{r^3}, \quad (10)$$

$$\frac{\partial J}{\partial t} + V_r \frac{\partial J}{\partial r} = \frac{r B_z B_{\phi s}}{2\pi \Sigma}, \quad (11)$$

$$\frac{\Sigma c_s^2}{2H} = \frac{\pi}{2} G \Sigma^2 + \frac{GM_c \Sigma H}{4r^3} + \frac{1}{8\pi} (B_{rs}^2 + B_{\phi s}^2), \quad (12)$$

and

$$\frac{H}{2\pi} \frac{\partial \Psi}{\partial t} = -r H V_r B_z - \eta B_{rs} - \frac{r \eta_H}{B} B_z B_{\phi s} - \frac{r \eta_A}{B^2} B_{rs} B_z^2; \quad (13)$$

Σ is the column density, defined as $\Sigma = 2H\rho$, and M_c is the mass of the central star. These equations are a simplified set of those used in Braiding & Wardle (2012), removing the $H \partial B_z / \partial r$ terms that were used to refine the structure of the disc and shocks in the full similarity solutions.

The Ohmic and ambipolar diffusion terms scale together, to a zeroth-order approximation, as they possess a similar dependence upon B and appear in the induction equation multiplied with the same field component. Ohmic diffusion is expected to be more important in the inner regions where the density is high, while ambipolar diffusion dominates in the outer regions (Wardle 2007). As the field within the disc is effectively vertical, ambipolar and Ohmic diffusion influence the field drift in the same direction, and only one term is required to study the change in disc behaviour introduced by Hall diffusion. We then describe ambipolar and Ohmic diffusion using the Pedersen diffusivity,

$$\eta_P = \eta_A + \eta, \quad (14)$$

which we parameterise by the constant nondimensional Pedersen diffusion parameter, $\tilde{\eta}_P$ (described in Appendix A).

Similarly, we define a nondimensional Hall diffusion parameter $\tilde{\eta}_H$ such that the Hall coefficient η_H scales with the surface density and thickness of the thin disc in a similar way to the Pedersen coefficient (see Appendix A). The direction of Hall diffusion depends upon the product $\eta_H(\mathbf{B} \cdot \boldsymbol{\Omega})$, where the sign of η_H depends on microphysics within the disc (Wardle & Ng 1999); in our simulations we vary the sign of $\tilde{\eta}_H$ in order to examine the effect of reversing the magnetic field. By studying the range of parameters that give stable accretion flow solutions it may be possible to place constraints on their physical values in observed accretion systems.

We look for solutions to the equations that take the form

of power laws with respect to a similarity variable $x = r/c_s t$ in the limit that $x \rightarrow 0$. These may then be thought of as valid in the limits $r \rightarrow 0$ or $t \rightarrow \infty$, that is, the innermost regions or late stages of the accretion flow. Only two physical solutions exist to these equations (see Appendix A): a secularly-evolving Keplerian disc and a magnetically-diluted near-free fall collapse. We present these in the following two sections.

3 KEPLERIAN DISC SOLUTION

The first solution is a Keplerian disc, where the material is supported against gravity by rotation ($V_\phi = \pm\sqrt{GM/r}$; the \mathbf{z} -axis is defined such that B_z is always positive) and accretion onto the central mass is slow. The disc is described by the simplified set of equations:

$$g_r + \frac{J^2}{r^3} = 0, \quad (15)$$

$$\frac{\partial J}{\partial r} = \frac{r B_{\phi s} B_z}{2\pi \Sigma V_r}, \quad (16)$$

$$H V_r + \frac{\eta_H}{B} B_{\phi s} + \frac{\eta_P}{B} B_{rs} = 0, \quad (17)$$

$$\frac{GM_c}{2r^3} H^2 + \pi G \Sigma H - c_s^2 = 0, \quad (18)$$

$$\dot{M} = \text{constant}, \quad (19)$$

$$J = r V_\phi, \quad (20)$$

$$B_{rs} = \frac{4}{3} B_z, \quad (21)$$

$$B_{\phi s} = -\delta B_z, \quad (22)$$

$$\text{and } \Psi = \frac{8}{3} \pi r^2 B_z; \quad (23)$$

the other terms in Equations 6–13, particularly the derivatives with respect to time, are negligible in this limit. The induction equation (17) can be written quite simply as $V_r + V_{Br} = 0$, that is, the inward radial velocity of the fluid is balanced by the drift of the magnetic field with respect to the gas (V_{Br}). Any accretion through the centrifugally-supported disc is regulated by the outward diffusion of the magnetic field against the flow; the magnetic diffusion then determines the column density Σ . The accretion rate \dot{M} and the direction of rotation in the disc are determined by the magnetic torque $B_{\phi s} B_z = -\delta B_z^2$, and are not affected by a global reversal of the magnetic field.

Hall diffusion can either add to or act against the Pedersen diffusion in the disc, as η_H can be either positive or negative, while η and η_A are both positive. The cap on the azimuthal field, δ , can have either sign, creating counter rotating solutions when δ is negative. Hall diffusion acts in the opposite direction in the counter rotating solutions (as the Hall term in Equation 17 has the opposite sign), causing there to be symmetry between the clockwise solutions with η_H positive and large and the anticlockwise solutions with large negative η_H .

The power law solution is given by:

$$M = \frac{\dot{m} c_s^3}{G} t, \quad \dot{M} = \frac{\dot{m} c_s^3}{G}, \quad (24)$$

$$V_r = -\frac{\dot{m}}{\sigma_1} \left(\frac{r c_s}{t} \right)^{1/2}, \quad \Sigma = \frac{\sigma_1 c_s^{5/2}}{2\pi G} \frac{t^{1/2}}{r^{3/2}}, \quad (25)$$

$$V_\phi = \pm \sqrt{\frac{\dot{m} c_s^3 t}{r}}, \quad J = \pm \sqrt{\dot{m} c_s^3 r t}, \quad (26)$$

$$B_z = \frac{(\dot{m} c_s^3)^{3/4}}{\sqrt{2|\delta|G}} \frac{t^{1/4}}{r^{5/4}}, \quad \Psi = \frac{8\pi(\dot{m} c_s^3)^{3/4}}{3\sqrt{2|\delta|G}} r^{3/4} t^{1/4}, \quad (27)$$

$$H = h_1 \frac{r^{3/2}}{\sqrt{c_s t}}, \quad (28)$$

where the nondimensional constants characterising the collapse are \dot{m} , the nondimensional infall rate onto the central star, δ , the artificial cap placed on the azimuthal magnetic field component, and $\tilde{\eta}_H$ and $\tilde{\eta}_P$, the Hall and Pedersen diffusion coefficients. The sign of J and V_ϕ matches that of δ , as the azimuthal drift of the fluid against rotation causes the field lines to bend at the disc surface. The magnitude of the cap on the magnetic braking, not its direction, influences the vertical field and the surface density. Finally, σ_1 and h_1 are constants, determined by the expressions

$$h_1 = \sqrt{\frac{2}{\dot{m}[1 + (f/2\delta)^2]}} \quad (29)$$

$$\text{and } \sigma_1 = \frac{\sqrt{2\dot{m}} f}{2|\delta|\sqrt{(2\delta/f)^2 + 1}} \quad (30)$$

(from Equations 17 and 18), where f is given by the equation

$$f = \frac{4}{3} \tilde{\eta}_P - \delta \tilde{\eta}_H \sqrt{\frac{25}{9} + \delta^2}. \quad (31)$$

This coefficient must be positive for accretion to occur through the disc — the opposite sign corresponds to a solution in which the field is diffusing inwards against the outwards flow of matter. For the typical value of $\delta = 1$, the condition $f \geq 1$ becomes

$$\tilde{\eta}_P \geq \tilde{\eta}_H \sqrt{\frac{17}{8}}, \quad (32)$$

creating a forbidden region of parameter space, shaded in Fig. 1 and bound by the solid line $f = \Sigma = 0$.

Fig. 1 illustrates the two-dimensional region of magnetic diffusion parameter space that gives feasible values of the surface density Σ at $r = 1$ au when $t = 10^4$ years, $c_s = 0.19$ km s⁻¹ and $\delta = 1$ for solutions with an accretion rate of $\dot{M}_c = 10^{-5}$ M_⊙ yr⁻¹ (corresponding to $B_z = 1.15$ G). The dotted line represents a surface density at 1 au of $\Sigma = 1700$ g cm⁻²; this is the value from the minimum mass solar nebula model (Weidenschilling 1977) in which the surface density of the solar nebula is estimated by adding sufficient hydrogen and helium to the solid bodies in the solar system to recover standard interstellar abundances, and spreading this material smoothly into a disc. The dashed lines correspond to higher surface densities that are more like those expected to occur in protostellar discs.

The radial scaling of the surface density ($\Sigma \propto r^{-3/2}$) in this solution is that expected from the minimum mass solar nebular (Weidenschilling 1977) and other simulations of protostellar discs (e.g. Tsuribe 1999; Vorobyov & Basu 2009); and the magnetic field scaling also matches that from theory, particularly of discs that support disc winds ($B_z \propto r^{-5/4}$; Blandford & Payne 1982). The azimuthal field component blows up with respect to the other field components in this small x limit because the azimuthal magnetic field drift speed is slow compared to the Keplerian speed. The model

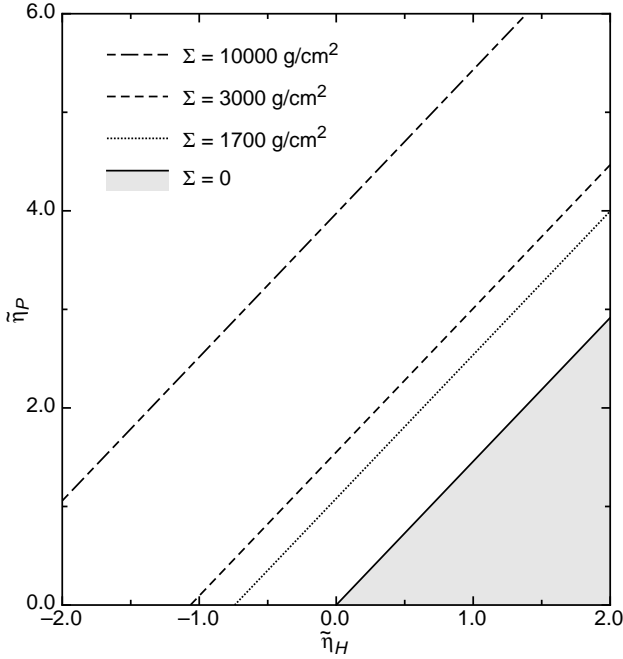


Figure 1. Exploration of the relationship between the Hall and Pedersen diffusivities $\tilde{\eta}_H$ and $\tilde{\eta}_P$ in the Keplerian disc solution for various disc surface densities Σ at $r = 1$ au and $t = 10^4$ yr, when the azimuthal field cap $\delta = 1$, the sound speed $c_s = 0.19$ km s $^{-1}$ and the accretion rate is $\dot{M} = 10^{-5} M_\odot$ yr $^{-1}$ (corresponding to $B_z = 1.15$ G). The solid black line is $\Sigma = 0$, and the shaded region beneath has no solutions describing an accreting disc rotating in the clockwise direction. The dashed lines correspond to $\Sigma = 1700, 3000$ and 10000 g cm $^{-2}$ respectively; the equivalent counter rotating solutions are reflected about $\tilde{\eta}_H = 0$.

adopted for the vertical angular momentum transport is unable to properly account for the effects of magnetic braking in the small x limit, so $B_{\phi s}$ must be capped. If a different scaling for η_H were adopted then Hall diffusion could act to limit $B_{\phi s}$ such that the cap becomes unnecessary; this should be examined in future disc studies.

As is clear from Fig. 1, there is no solution describing a disc with purely Hall diffusion where the Hall diffusion parameter is positive, as the positive Hall parameter acts to restrain the effects of ambipolar and Ohmic diffusion when the disc is rotating in the clockwise direction. However, when the Hall diffusion parameter is negative (corresponding to a reversal of the magnetic field) it acts in the same radial direction as the Pedersen diffusion, enhancing the radial diffusion of the magnetic field. When the disc is counter rotating, the opposite holds true, so that the solutions in Fig. 1 are reflected across the vertical line $\tilde{\eta}_H = 0$. In the case of pure ambipolar diffusion the field moves inward slower than the neutral particles and the solution reduces to the fiducial solution of KK02 (their equations 51–57).

Fig. 2 plots Σ against B_z and \dot{M} , showing the influence of Hall diffusion on Σ at 1 au of a disc with $\tilde{\eta}_P = \delta = 1$. The surface density increases with the negative Hall parameter to create heavier discs, while a positive value of $\tilde{\eta}_H$ causes Σ to drop off dramatically as $\tilde{\eta}_H$ approaches the limit of $f \geq 0$. Once more the opposite behaviour holds true when the disc is counter rotating.

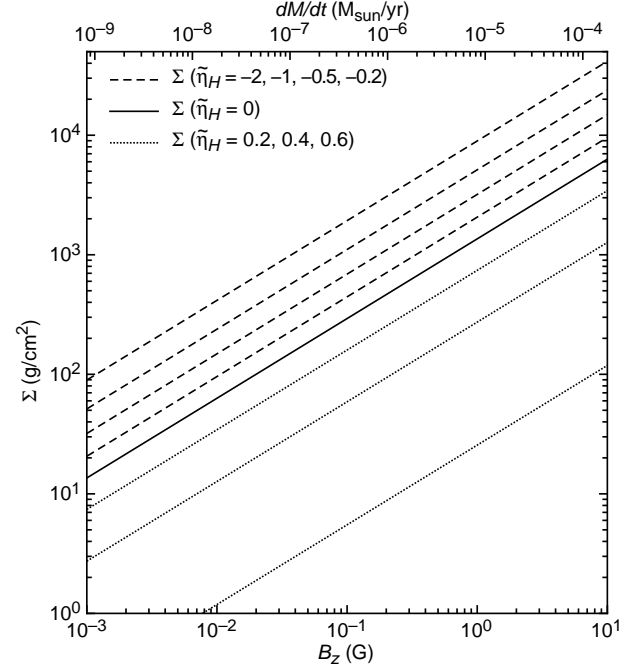


Figure 2. Disc surface density Σ against the vertical magnetic field component B_z and central mass accretion rate \dot{M} for different values of the Hall diffusion parameter $\tilde{\eta}_H$ for a Keplerian disc rotating in the clockwise direction with Pedersen parameter $\tilde{\eta}_P = \delta = 1.0$ and $c = 0.19$ km s $^{-1}$, at $r = 1$ au and $t = 10000$ yr. The solid line corresponds to $\tilde{\eta}_H = 0$; the dashed lines above are $\tilde{\eta}_H = -0.2, -0.5, -1.0$ and -2 respectively; and the dotted lower lines are $\tilde{\eta}_H = 0.2, 0.4$ and 0.6 .

Equations 15–20 may be written in a more physical form by writing the variables as functions of B_z , r , t , the diffusion parameter f and the azimuthal field cap. The magnetic field components are then given by Equations 21–23, while the other variables are:

$$\dot{M} = \left(\frac{(2\delta)^2 B_z^4}{Gt} r^5 \right)^{1/3}, \quad (33)$$

$$M = \left(\frac{(2\delta t)^2 B_z^4}{G} r^5 \right)^{1/3}, \quad (34)$$

$$\Sigma = \frac{c_s f}{2\pi |\delta|} \sqrt{\frac{M}{2G[(2\delta/f)^2 + 1]}} r^{-3/2}, \quad (35)$$

$$H = \sqrt{\frac{2}{G[1 + (f/2\delta)^2] M}} c_s r^{3/2}, \quad (36)$$

$$J = (2\delta G B_z^2 r^4 t)^{1/3}, \quad (37)$$

$$V_\phi = (2\delta G B_z^2 r t)^{1/3} \quad (38)$$

$$\text{and } V_r = -\frac{\delta B_z^2 r}{\pi V_\phi \Sigma}. \quad (39)$$

This model description shows more clearly that the direction of rotation in the disc is tied to the direction of magnetic braking and emphasizes the build up of the central mass occurs through accretion against magnetic field diffusion.

Alternatively, the disc variables can be thought of as functions of M and \dot{M} , by treating t as the ratio of M to \dot{M} rather than the age of the system. The solution can then be regarded as a steady state Keplerian disc, where the vari-

ables are defined by

$$B_z = \left(\frac{GM\dot{M}^2}{(2\delta)^2 r^5} \right)^{1/4}, \quad (40)$$

$$J = \sqrt{GM\dot{M}r}, \quad (41)$$

$$V_\phi = V_K = \sqrt{\frac{GM}{r}} \quad (42)$$

$$\text{and } V_r = -\frac{\dot{M}}{2\pi\Sigma r}; \quad (43)$$

Σ and H are unchanged from Equations 35 and 36, and the field components are as in Equations 21–23.

The amount of Hall diffusion in the flow helps determine how much matter is accreted onto the central object compared with that remaining in the disc; decreasing the Hall parameter $\tilde{\eta}_H$ in comparison to a constant $\tilde{\eta}_P$ causes the surface density become large as the radial field diffusion slows the rate of infall. The ambipolar diffusion-only fiducial solution of KK02 has an accretion rate of $\dot{M}_c = 7.66 \times 10^{-6} M_\odot \text{ yr}^{-1}$, and the surface density at 1 au is 1310 g cm^{-2} . Adding a positive value of $\tilde{\eta}_H$ to this solution causes the surface density to decrease, which is problematic if one expects to form large planets in a protostellar disc, however, if the Hall parameter were negative then the surface density would be raised to a more realistic value. This again is only true for solutions rotating in the clockwise direction; however, as the sign of η_H depends on microphysics in the disc, the general point that planetary system architecture depends on the product $\eta_H(\mathbf{B} \cdot \Omega)$ stands.

4 FREE FALL SOLUTION

The second similarity solution describes the infall when the magnetic braking is very efficient at removing angular momentum from the flow. In this case there is very little angular momentum remaining and the reduced centrifugal support inhibits disc formation, so that the collapsing flow becomes a supersonic magnetically-diluted free fall onto the central protostellar mass. This solution is representative of the magnetic braking catastrophe that affects many numerical simulations of gravitational collapse. The collapse is described by the equations:

$$V_r \frac{\partial V_r}{\partial r} = g_r, \quad (44)$$

$$\frac{\partial J}{\partial r} = \frac{r B_{\phi s} B_z}{2\pi \Sigma V_r}, \quad (45)$$

$$V_r H + \frac{\eta_H}{B} B_{\phi s} + \frac{\eta_P}{B^2} B_{rs} B_z = 0, \quad (46)$$

$$\frac{GM}{2r^3} H^2 + \frac{(B_{rs}^2 + B_{\phi s}^2)}{4\pi \Sigma} H - c_s^2 = 0, \quad (47)$$

$$\dot{M} = \text{constant} \quad (48)$$

$$J = r V_\phi, \quad (49)$$

$$B_{rs} = B_z \quad (50)$$

$$\Psi = 2\pi r^2 B_z \quad (51)$$

$$\text{and } B_{\phi s} = -\min \left[\frac{-\eta_H}{\eta_P} B; \delta B_z \right]; \quad (52)$$

as in the Keplerian disc solution the induction equation takes the simplified form $V_r + V_{Br} = 0$, however here the magnetic diffusion is assumed to be large and dynamically important.

In this similarity solution any rotation remaining in the flow is that induced by the magnetic “braking”, which can cause rotation by Hall diffusion of the field lines tied to the electrons in the azimuthal direction, which creates a rotational torque on the neutrals and grains as they fall inward rapidly.

The dimensional form of this similarity solution is given by the complete set of fluid variables:

$$M = \frac{\dot{m} c_s^3}{G} t, \quad \dot{M} = \frac{\dot{m} c_s^3}{G} \quad (53)$$

$$V_r = -\sqrt{\frac{2\dot{m} c_s^3 t}{r}}, \quad \Sigma = \frac{1}{\pi G} \sqrt{\frac{\dot{m} c_s^3}{2^3 r t}}, \quad (54)$$

$$V_\phi = j_{1,2} c_s, \quad J = j_{1,2} c_s r, \quad (55)$$

$$B_z = \frac{c_s^2 b_{z1,2}}{G^{1/2}} r^{-1}, \quad \Psi = \frac{2\pi c_s^2 b_{z1,2}}{G^{1/2}} r, \quad (56)$$

$$B_{rs} = B_z, \quad H = h_{1,2} \frac{r^{3/2}}{\sqrt{c_s t}}, \quad (57)$$

and

$$B_{\phi s} = -\min \left[\frac{-\tilde{\eta}_H}{\tilde{\eta}_P} B; \delta B_z \right]. \quad (58)$$

The sign of the Hall diffusion coefficient and the cap on $B_{\phi s}$ must be taken into account when calculating the minimum of Equation 58, as this should be an absolute minimum. We define the $\hat{\mathbf{z}}$ direction such that B_z is always positive, and as the radial velocity is negative then conservation of angular momentum (Equation 49) requires $B_{\phi s}$ and J to have the opposite sign. Any twisting of the field by Hall diffusion is balanced by spinning the fluid in the opposite direction.

Given that there are two possible values for $B_{\phi s}$, depending upon the chosen values of the parameters $\tilde{\eta}_{H,P}$ and δ , there are two possible coefficients for B_z , J and H . These are referenced by the subscripts 1 and 2 depending on which term in Equation 58 is smaller. The first set of coefficients occur when

$$\sqrt{\frac{2\tilde{\eta}_H^2}{\tilde{\eta}_P^2 - \tilde{\eta}_H^2}} < |\delta|, \quad (59)$$

so that the azimuthal field is given by

$$B_{\phi s} = \frac{\sqrt{2}\tilde{\eta}_H B_z}{\sqrt{\tilde{\eta}_P^2 - \tilde{\eta}_H^2}}. \quad (60)$$

The vertical field coefficient b_{z1} is then the single real root of the polynomial

$$b_{z1}^8 - \frac{\dot{m}^2}{2\tilde{\eta}_P^2 f_1} b_{z1}^6 - \frac{\dot{m}^6}{4\tilde{\eta}_P^4 f_1^4} = 0, \quad (61)$$

where f_1 is a constant defined by the magnetic diffusion parameters:

$$f_1 = \frac{\tilde{\eta}_P^2 + \tilde{\eta}_H^2}{\tilde{\eta}_P^2 - \tilde{\eta}_H^2}. \quad (62)$$

The coefficients for j and h are given by

$$j_1 = -\frac{b_{z1}^2}{\dot{m}} \frac{\sqrt{2}\tilde{\eta}_H}{\sqrt{\tilde{\eta}_P^2 - \tilde{\eta}_H^2}} \quad (63)$$

and

$$h_1 = \frac{f_1 b_{z1}^2}{\sqrt{2}\dot{m}^3} \left[-1 + \sqrt{1 - \frac{4\dot{m}^2}{f_1^2 b_{z1}^4}} \right]. \quad (64)$$

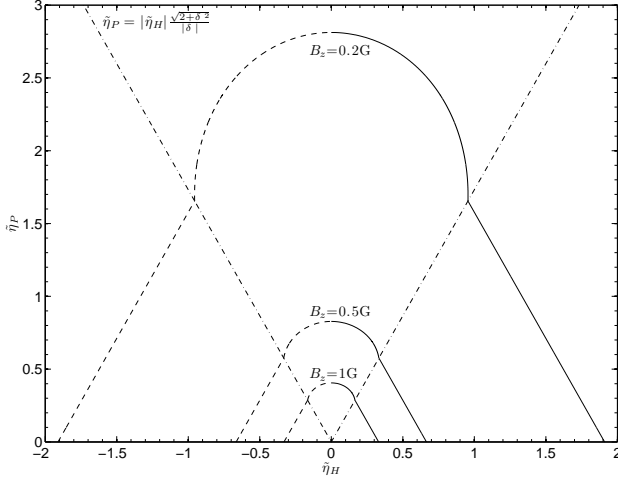


Figure 3. The relationship between the Hall and Pedersen diffusion parameters $\tilde{\eta}_H$ and $\tilde{\eta}_P$ in the free fall solution, for various midplane magnetic field strengths B_z at $r = 1$ au and $t = 10^4$ yr, where the azimuthal field cap $\delta = 1$ and the sound speed $c_s = 0.19$ km s $^{-1}$. We take the central mass accretion rate to be $\dot{M} = 10^{-5}$ M $_{\odot}$ yr $^{-1}$ (corresponding to a surface density at $r = 1$ au of $\Sigma = 5.03$ g cm $^{-2}$). The solid curves correspond to negative rotation, the dashed ones positive, and the boundary between the two values of $B_{\phi s}$ is the dot-dashed line $\tilde{\eta}_P = \tilde{\eta}_H \sqrt{2 + \delta^2} / \delta$.

When there is only ambipolar diffusion, this solution reduces to the nondimensional asymptotic behaviour of the “strong braking” collapse of KK02 (their equations 66–71). In their solution $f_1 = 1$ and $j_1 \approx 0$, so that the angular momentum and the azimuthal field component are reduced to a small plateau value as the strong magnetic braking removes almost all of the angular momentum early in the collapse (see their fig. 10). Their similarity solutions tended towards this behaviour when the magnetic braking parameter α was large, even though it does not appear in the coefficients of the power law solutions.

This set of coefficients corresponds to the curves in Fig. 3, which plots $\tilde{\eta}_P$ against $\tilde{\eta}_H$ for cores with varied magnetic fields and $\dot{M} = 10^{-5}$ M $_{\odot}$ yr $^{-1}$ at 1 au and $t = 10^4$ yr. The direction of rotation changes with the sign of $\tilde{\eta}_H$, with the solid lines denoting negative rotation and the dotted lines positive. As the field increases the size of the area enclosed by the curve decreases, and the solutions tend towards the asymptote $\tilde{\eta}_P = |\tilde{\eta}_H|$ as both the diffusivities tend to zero. The $\tilde{\eta}_P$ -intercept occurs at

$$\tilde{\eta}_P = \frac{\dot{m}}{2b_{z1}} \left[1 + \sqrt{1 + \frac{4\dot{m}^2}{b_{z1}^4}} \right]^{1/2}, \quad (65)$$

corresponding to the value of $\tilde{\eta}_P$ needed to give the same magnetic field strength in KK02. The solid line intercepting the curves is $\tilde{\eta}_P = |\tilde{\eta}_H| \sqrt{2 + \delta^2} / \delta$, which marks the boundary between the coefficient sets and in the limit of large $|\delta|$ also tends towards $\tilde{\eta}_P = |\tilde{\eta}_H|$.

The second set of coefficients apply when the inequality

$$\sqrt{\frac{2\tilde{\eta}_H^2}{\tilde{\eta}_P^2 - \tilde{\eta}_H^2}} > |\delta| \quad (66)$$

is satisfied, so that $B_{\phi s}$ takes on the second value in Equation 58, $B_{\phi s} = -\delta B_z$. The coefficient b_{z2} is given by the single

positive real root of the equation

$$b_{z2}^8 - \frac{\dot{m}^2(1 + \delta^2)}{2f_2^2} b_{z2}^6 - \frac{\dot{m}^6}{4f_2^4} = 0 \quad (67)$$

where f_2 is

$$f_2 = \tilde{\eta}_P - \tilde{\eta}_H \delta \sqrt{2 + \delta^2}. \quad (68)$$

This then gives the other coefficients as

$$j_2 = -\frac{\delta b_{z2}^2}{\dot{m}} \quad (69)$$

and

$$h_2 = \frac{(1 + \delta^2)b_{z2}^2}{\sqrt{2\dot{m}^3}} \left[-1 + \sqrt{1 + \frac{4\dot{m}^2}{(1 + \delta^2)^2 b_{z2}^4}} \right]. \quad (70)$$

This solution was not explored in KK02, as without Hall diffusion the left term in Equation 58 is zero.

The second set of coefficients corresponds to the straight lines outside of the curves in Fig. 3, again with negative rotation drawn as solid lines and positive as dashed. The slope of the line (and the direction of rotation) are both dependant upon the sign of δ , i.e. the cap on $B_{\phi s}$; in this solution the cap may have either sign, corresponding to the direction in which the field was azimuthally bending from the vertical before the cap was attained. Under a global reversal of the field the infalling matter spins in the opposite direction.

The equations may also be rearranged to give the variables as functions of the surface density and magnetic field:

$$V_r = -\sqrt{\frac{2GM}{r}}, \quad (71)$$

$$M = -2\pi r \Sigma V_r t \quad (72)$$

$$\dot{M} = -2\pi r \Sigma V_r \quad (73)$$

$$\text{and } J = \frac{B_{\phi s} B_z}{2\pi \Sigma V_r} r^2, \quad (74)$$

where the field components are defined in Equations 50–52 and the scale height of the disc is given by the equations:

$$H_1 = \frac{f_1 B_z^2 r^{7/2} t}{\sqrt{2GM^3}} \left[-1 + \sqrt{1 - \frac{4c_s^2 M^2}{f_1^2 B_z^4 r^4 t^2}} \right] \text{ or} \quad (75)$$

$$H_2 = \frac{(1 + \delta^2) B_z^2 r^{7/2} t}{\sqrt{2GM^3}} \left[-1 + \sqrt{1 - \frac{4c_s^2 M^2}{(1 + \delta^2)^2 B_z^4 r^4 t^2}} \right] \quad (76)$$

depending on which value of $B_{\phi s}$ is adopted.

This similarity solution exemplifies the magnetic braking catastrophe that occurs in numerical simulations of star formation when the magnetic braking of a collapsing core is so strong that all angular momentum is removed from the gas and it is impossible to form a rotationally-supported disc (e.g. Mellon & Li 2008, 2009). It is clear from Equations 55b, 58 and 63 that when there is no Hall diffusion the magnetic braking causes $B_{\phi s} = J = 0$, which would prevent Keplerian disc formation.

The introduction of Hall diffusion to the similarity solution can cause additional twisting of the field lines and magnetic braking, or it can cause a reduction in magnetic braking by twisting the magnetic field lines in the opposite direction, in effect spinning up the collapse. The direction of the Hall diffusion depends upon the orientation of the field

with respect to the axis of rotation, and it is obvious that this directionality has an important effect on the magnetic braking catastrophe. The linear scaling of the angular momentum with radius (Equation 55) suggests that the point mass at the origin has no angular momentum, however, Hall diffusion will likely ensure that the angular momentum shall drop to a plateau value similar to that in the ambipolar diffusion-only collapse of KK02's fig. 10 in full simulations of such accretion flows.

5 DISCUSSION

The two similarity solutions represent both sides of the magnetic braking catastrophe of star formation. In the first solution, the magnetic braking is limited and a rotationally-supported disc similar to those in the simulations of Machida et al. (2011) forms, while in the second no disc forms as catastrophic magnetic braking removes almost all of the angular momentum as in the simulations of Mellon & Li (2009). The magnetic diffusion, in particular Hall diffusion, is clearly important in determining whether a disc forms, its size and rotational behaviour and the density profile within the disc.

This can have wide implications on planet formation, for the size and surface density of the disc will then depend upon the product $\eta_H(\mathbf{B} \cdot \Omega)$, where the sign of η_H depends upon the abundances of charged grains and electrons relative to the neutrals in the disc (Wardle & Ng 1999). This means that the outcome of planet formation in two otherwise identical discs can be vastly different if they are initially rotating in opposite directions. Our results in §3 show that one does not require a large amount of Hall diffusion for this asymmetry to be observable in the surface density of a protostellar disc.

Our second solution shows a similar asymmetry caused by Hall diffusion, as all of the initial angular momentum has been removed from the collapse by magnetic braking and the only rotation of the flow is that induced by Hall diffusion. The direction of rotation corresponds directly with the sign of the Hall parameter, however the angular momentum remains dynamically unimportant. This solution ought to apply in simulations of accretion flows where the magnetic braking parameter α and the cap on the azimuthal field δ are large, as in the “strong braking” star formation similarity solution of KK02.

The prescription for the magnetic braking used in this simple model is limited by our use of a cap placed on $B_{\phi s}$ to account for missing physics such as non-axisymmetric effects (e.g. the magnetorotational instability; Sano & Stone 2002). The azimuthal field takes the value of this cap in our Keplerian disc solution, and also in the free fall solution when the Hall diffusivity is large, however the value $\delta = 1$ adopted in our plots is that expected from a disc with a disc wind. A more complete description of the angular momentum transport would include the calculation of a disc wind (Blandford & Payne 1982), as these are observed in simulations of protostellar and other accretion discs (e.g. Machida et al. 2007; Mellon & Li 2009).

The similarity solutions of Braiding & Wardle (2012) all demonstrated Keplerian disc formation in the inner region of the protostellar collapse of a molecular cloud core,

matching onto the power law solution described in §3 at the inner boundary. Enforcing disc formation, Hall diffusion was shown to affect the mass of the protostar and accretion disc, as well as the size and structure of the disc. The free fall solution has yet to be adopted as a boundary condition in the full model, but its applicability to collapsing cores must be studied in future to show the importance of Hall diffusion in spinning up the flow, facilitating disc formation in order to solve the magnetic braking catastrophe.

6 CONCLUSIONS

We have presented two distinct power law similarity solutions to the MHD equations describing a flattened accretion flow with magnetic diffusion. The first of these represented a rotationally-supported disc through which gas is slowly accreted while the magnetic field diffuses outwards against the flow. The second was a free fall collapse onto the star in which almost all of the angular momentum has been removed from the fluid and no rotationally-supported disc may form. These have been used to show that even a small amount of Hall diffusion can dramatically change the dynamics of gravitational collapse.

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APPENDIX A: SELF-SIMILAR EQUATIONS

Gravitational collapse occurs over many orders of magnitude in density, so that the point mass has negligible dimensions in comparison with the accretion flow. The self-similarity of the wave of infall, which propagates outwards at the speed of sound, arises because of the lack of characteristic time and length scales. Then the only dimensional quantities are the gravitational constant G , the isothermal sound speed c_s , the local radius r and the instantaneous time t , so that, except for scaling factors, all quantities must be functions of the similarity variable x , defined as:

$$x = \frac{r}{c_s t}. \quad (\text{A1})$$

We then define a set of nondimensional fluid variables that depend only upon x :

$$\Sigma(r, t) = \left(\frac{c_s}{2\pi G t}\right) \sigma(x), \quad g_r(r, t) = \left(\frac{c_s}{t}\right) g(x), \quad (\text{A2})$$

$$V_r(r, t) = c_s u(x), \quad H(r, t) = c_s t h(x), \quad (\text{A3})$$

$$V_\phi(r, t) = c_s v(x), \quad J(r, t) = c_s^2 t j(x), \quad (\text{A4})$$

$$M(r, t) = \left(\frac{c_s^3 t}{G}\right) m(x), \quad \dot{M}(r, t) = \left(\frac{c_s^3}{G}\right) \dot{m}(x), \quad (\text{A5})$$

$$\mathbf{B}(r, t) = \left(\frac{c_s}{G^{1/2} t}\right) \mathbf{b}(x), \quad \Psi(r, t) = \left(\frac{2\pi c_s^3 t}{G^{1/2}}\right) \psi(x). \quad (\text{A6})$$

In the thin disc approximation used here, the diffusivities are assumed to be constant with height within the disc, and the Ohmic and ambipolar diffusion terms scale together to a zeroth-order approximation. They are combined into the Pedersen diffusivity, which scales within the disc as

$$\eta_P = \tilde{\eta}_P \frac{c^2 t b^2 h^{3/2}}{\sigma^{3/2}}; \quad (\text{A7})$$

the self-similarity of the diffusivity depends upon the implicit relationship $\rho \propto \rho_i$, which holds true across a large range of densities (Kamaya & Nishi 2000). We adopt a similar scaling for the Hall diffusion coefficient η_H as a matter of pragmatism:

$$\eta_H = \tilde{\eta}_H \frac{c^2 t b^2 h^{3/2}}{\sigma^{3/2}}, \quad (\text{A8})$$

where $\tilde{\eta}_H$ is the nondimensional Hall diffusion parameter used to characterise the solutions, the ratio of the ambipolar to Hall terms becomes the most important factor in describing the magnetic behaviour of the solutions.

For convenience we define $w \equiv x - u$ and then use the similarity variables to write Equations 9–13 as:

$$\frac{dm}{dx} = x\sigma, \quad (\text{A9})$$

$$(1 - w^2) \frac{1}{\sigma} \frac{d\sigma}{dx} = g + \frac{b_{r,s} b_z}{\sigma} + \frac{j^2}{x^3} + \frac{w^2}{x}, \quad (\text{A10})$$

$$\frac{dj}{dx} = \frac{1}{w} \left(j - \frac{x b_z b_{\phi,s}}{\sigma} \right), \quad (\text{A11})$$

$$\frac{\sigma \dot{m}}{x^3} h^2 + (b_{r,s}^2 + b_{\phi,s}^2 + \sigma^2) h - 2\sigma = 0, \quad (\text{A12})$$

$$\psi - x w b_z + (\tilde{\eta}_H b_{\phi,s} b + \tilde{\eta}_P b_{r,s} b_z) x b_z h^{1/2} \sigma^{-3/2} = 0, \quad (\text{A13})$$

$$\text{and } \frac{d\psi}{dx} = x b_z. \quad (\text{A14})$$

These are augmented by the supplementary definitions:

$$m = x w \sigma, \quad (\text{A15})$$

$$\dot{m} = -x u \sigma, \quad (\text{A16})$$

$$\text{and } g = -\frac{m}{x^2}, \quad (\text{A17})$$

while the field components are given by

$$b_{r,s} = \frac{\psi}{x^2}, \quad (\text{A18})$$

$$b_{\phi,s} = -\min \left[\frac{2\alpha\psi}{x^2} \left[\frac{j}{x} - \frac{\tilde{\eta}_H h^{1/2} b}{\sigma^{3/2}} \left(b_{r,s} - h \frac{db_z}{dx} \right) \right] \right. \\ \left. \left[1 + \frac{2\alpha\tilde{\eta}_P h^{1/2} \psi b_z}{x^2 \sigma^{3/2}} \right]^{-1}; \delta b_z \right]. \quad (\text{A19})$$

We find the similarity solutions by assuming that the fluid variables take the form of power law relations in x :

$$\sigma = \sigma_1 x^{-p} \quad (\text{A20})$$

$$b_z = b_{z1} x^{-q} \quad (\text{A21})$$

$$j = j_1 x^{-r} \quad (\text{A22})$$

where p , q , and r are real numbers. We substitute these into the self-similar equations, and by taking the limit as $x \rightarrow 0$ we find the dominant terms and solve for the exponents and coefficients of the power law relations. There are only two nontrivial solutions (derived in Braiding 2011): a Keplerian disc with $p = 3/2$, $q = 5/4$ and $r = -1/2$; and the near-free fall collapse with $p = 1/2$, $q = 1$ and $r = -1$.

A1 Keplerian disc solution

The first solution is the Keplerian disc presented in §3, where the mass is supported against gravity by its angular momentum. The similarity solution in nondimensional form is:

$$m = \dot{m}, \quad \sigma = \sigma_1 x^{-3/2}, \quad (\text{A23})$$

$$u = -\frac{\dot{m}}{\sigma_1} x^{1/2}, \quad j = \pm \sqrt{\dot{m} x}, \quad (\text{A24})$$

$$\psi = \frac{4}{3} x^2 b_z, \quad b_z = \frac{\dot{m}^{3/4}}{\sqrt{2|\delta|}} x^{-5/4}, \quad (\text{A25})$$

$$b_{r,s} = \frac{4}{3} b_z, \quad b_{\phi,s} = -\delta b_z, \quad (\text{A26})$$

$$h = h_1 x^{3/2}, \quad (\text{A27})$$

where the constants are defined as in §3. In the ambipolar diffusion limit these reduce to equations 51–57 of KK02.

A2 Free fall solution

The second solution is the set of relations that describe a free fall onto the central mass dominated by magnetic diffusion:

$$m = \dot{m}, \quad \sigma = \sqrt{\frac{\dot{m}}{2x}}, \quad (\text{A28})$$

$$u = -\sqrt{\frac{2\dot{m}}{x}}, \quad j = j_{1,2} x, \quad (\text{A29})$$

$$\psi = b_z x, \quad b_z = b_{z1,2} x^{-1}, \quad (\text{A30})$$

$$b_{r,s} = b_z, \quad b_{\phi,s} = -\min \left[\frac{\tilde{\eta}_H}{\tilde{\eta}_P} b; \delta b_z \right], \quad (\text{A31})$$

$$h = h_{1,2} x^{3/2}. \quad (\text{A32})$$

Given that there are two possible values for $b_{\phi,s}$, depending on the chosen values of the parameters $\tilde{\eta}_{H,A}$ and δ , there are two sets of coefficients for j , b_z and h . These are defined in §4 and referenced by the subscripts 1 and 2 depending on which term in Equation A31b they are associated with.

In the ambipolar diffusion limit this solution reduces to equations 66–71 of KK02, where $j = b_{\phi,s} \approx 0$.

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